Advanced ODE-Lecture 5
Global Existence and Comparison Principle

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Outline

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Motivation

- Existence of solutions on finite time intervals is not convenient in applications and in most cases we need existence on $[t_0, \infty)$ for any $(t_0, x_0) \in R \times R^n$. We need to know some often classes of systems with global existence.

- Comparison principle is an important tool to estimate bounds on the solutions without solving ODE.

- Comparison principle is variation of inequality techniques. So it can be regarded as an extension of inequality techniques.

Global Existence

1) Linear Boundedness

Definition 5.1 $f : R \times R^n \to R^n$ is **linearly bounded** if there exist $a \geq 0$ and $b \geq 0$ such that

$$\| f(t,x) \| \leq a \| x \| + b \quad \text{for all} \quad (t,x) \in R \times R^n .$$

Theorem 5.1 Suppose that $f(t,x)$ is continuous; locally Lipschitz on $R \times R^n$ and linearly bounded. Then the unique solution $x(t)$ of IVP has $I_{\max}^{+} = [t_0, \infty)$ for any $(t_0, x_0) \in R^{n+1}$.

Proof. Suppose that $I_{\max}^{+} = [t_0, \omega_+)$. We show $\omega_+ = \infty$. Since

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds ,$$

it is easy to see that

$$\| x(t) \| \le \| x_0 \| + \int_{t_0}^{t} \| f(s, x(s) \| ds \le \| x_0 \| + \int_{t_0}^{t} (a \| x(s) \| + b) ds$$

$$\le (\| x_0 \| + b(t - t_0)) + a \int_{t_0}^{t} \| x(s) \| ds .$$

If $\omega_+ < \infty$, then

$$\| x(t) \| \le (\| x_0 \| + b(\omega_+ - t_0)) + a \int_{t_0}^{t} \| x(s) \| ds$$

Application of Gronwall inequality yields

$$\| x(t) \| \le (\| x_0 \| + b(\omega_+ - t_0)) e^{a(\omega_+ - t_0)} < \infty .$$

This is contradicted by the extensibility theorem. It concludes that $\omega_+ = \infty$. □

Remark 5.1 Even for a linear system $x' = A(t)x + h(t)$, where $A(t)$, $h(t) \in C(R)$, not necessarily bounded on $(-\infty, \infty)$, so it is not necessarily linear bounded. Although the linearly bounded is a bit restrictive, it is easy to be checked.

2) Lyapunov-like Auxiliary Condition

Theorem 5.2 Let $f(t,x)$ be continuous and locally Lipschitz on $R\times R^n$. Suppose that there exist an auxiliary function $V(t,x): R\times R^n \to R$ of class C^1 such that

- $W_1(x) \le V(t,x) \le W_2(x)$ where $W_1(x)\ge 0$ with $W_1(x)=0 \Rightarrow x=0$;
  (Positive Definite)
- $\lim_{\|x\|\to\infty} W_1(x)=\infty$, **(Radially Unbounded)**;
- $\dot V(t,x) \overset{def.}{=} \dfrac{\partial V}{\partial t} + \dfrac{\partial V}{\partial x}\cdot f(t,x) \le a + bW_1(x)$.

Then the unique solution $x(t)$ of IVP has $I^+_{\max} = [t_0,\infty)$ for any $(t_0,x_0)\in R^{n+1}$.

Proof. By the above auxiliary conditions, we have

$$\frac{d}{dt}V(t,x(t)) = \frac{\partial V(t,x(t))}{\partial t} + \frac{\partial V(t,x(t))}{\partial x}\cdot f(t,x(t)) \le a + bW_1(x(t))$$

$$\le a + bV(t,x(t)).$$

Integrating this inequality yields

$$V(t,x(t)) \le V(t_0,x_0) + \int_{t_0}^{t} (a + bV(s,x(s))ds.$$

If $I_{\max}^{+}=[t_0,\omega_+)$ with $\omega_+<\infty$, then

$$V(t,x(t))\le\{V(t_0,x_0)+a(\omega_+-t_0)\}+b\int_{t_0}^{t}V(s,x(s))ds.$$

Application of Gronwall inequality gives the bound

$$V(t,x(t))\le\{a(\omega_+-t_0)+V(t_0,x_0)\}e^{b(t-t_0)},$$

which implies

$$W_1(x(t))\le\{a(\omega_+-t_0)+V(t_0,x_0)\}e^{b(\omega_+-t_0)}<\infty. \qquad \text{(F1)}$$

Meanwhile, it can be deduced that $\lim_{t\to\omega_+}\|x(t)\|=\infty$ by the extensibility theorem.

Then,

$$\lim_{t\to\omega_+}W_1(x(t))=\lim_{\|x\|\to\infty}W_1(x)=\infty.$$

This contradicts with (F1). So it shows that $\omega_+=\infty$. □

Corollary 5.1 Let $f(x)$ be locally Lipschitz on R^n. Suppose that there exist $V(x): R^n \to R$ of class C^1 such that

- $V(x) \geq 0$ with $V(x) = 0 \Rightarrow x = 0$; (**Positive Definite**)
- $\lim_{\|x\| \to \infty} V(x) = \infty$, (**Radially Unbounded**);
- $\dot{V}(x) \overset{def.}{=} \frac{\partial V}{\partial x} \cdot f(t,x) \leq a + bV(x)$.

Then the unique solution $x(t)$ of the IVP has $I_{\max}^{+} = [0, \infty)$ for any $x_0 \in R^n$.

Proof. Since $f(x)$ is free of t, then taking $V(x) = V(t,x) \equiv W_1(x) \equiv W_2(x)$ and $t_0 = 0$ obtains the result. □

Remark 5.2 How to find a desired Lyapunov-like candidate, there is no systematic way in general. It is still open in Math. However, the existence of Lyapunov-like function is guaranteed under some reasonable mild conditions. The details will be given in Lyapunov stability theory.

Systems with Global Existence

1) Gradient Systems

Suppose that $V(x): R^n \to R_{\geq 0}$ is a function of C^2. $x' = -\nabla V(x)$ is called a gradient system, where

$$\nabla V(x) = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots, \frac{\partial V}{\partial x_n} \right)^T = \left(\frac{\partial V}{\partial x} \right)^T.$$

Lemma 5.1 Suppose that $V(x): R^n \to R_{\geq 0}$ is a function of C^2 with $\lim_{\|x\| \to \infty} V(x) = \infty$. Then any solution of the gradient system exists for all $t \geq 0$.

Proof. Taking derivative along trajectories of the gradient system, we have

$$\frac{d}{dt} V(x(t)) = \frac{\partial V}{\partial x} \cdot x' = -\frac{\partial V}{\partial x} \cdot \left(\frac{\partial V}{\partial x} \right)^T = -\|\frac{\partial V(x)}{\partial x}\|^2 \leq 0.$$

This implies that $V(x(t)) \leq V(x_0)$ for all $t \geq 0$. Then we claim that $I_{\max} = [0, +\infty)$.

Otherwise there exists a time $\omega_+ < \infty$ s.t. $\overline{\lim_{t\to\omega_+^-}} \| x(t) \| = \infty$ by the extensibility theorem,. Then, there exists $\{t_n\} \to \omega_+^-$ as $n \to \infty$ s.t. $\lim_{\|x(t_n)\| \to \omega_+^-} V(x(t_n)) = \infty$. This contradicts with $V(x(t)) \le V(x_0)$. So $I_{\max} = [0, +\infty)$. This completes the proof. □

Remark 5.3 If $x \in V^{-1}(c) = \{x : V(x) = c\}$ is a regular point (i.e. $\nabla V(x) \neq 0$), the solution curve $x(t)$ is perpendicular to the level surface $V^{-1}(c)$. Since for any curve $\gamma(t) \in V^{-1}(c)$ with $\gamma(0) = x$ and $\gamma'(0) = y$, we have

$$0 = \frac{d}{dt} V(\gamma(t)) |_{t=0} = \frac{\partial V(\gamma(t))}{\partial x} \cdot \gamma'(t) |_{t=0} = \nabla V(x))^T \cdot y = \langle \nabla V(x), \quad y \rangle .$$

2) Hamiltonian Systems

Suppose that $H(x, y): R^n \times R^n \to R_{\geq 0}$ is a function of C^2.

$$x' = \nabla_y H(x, y); \quad y' = -\nabla_x H(x, y)$$

is called a Hamiltonian equation, where $H(x, y)$ is called a Hamiltonian function.

Since $f(x, y) = \left(\nabla_y H(x, y), \quad -\nabla_x H(x, y)\right)^T$ is locally Lipschitz by C^2, so the existence and uniqueness of solution is done. Suppose that $(x(t), y(t))$ is a solution. Then,

$$\frac{d}{dt} H(x(t), y(t)) = \nabla_x H(x(t), y(t))\, x'(t) + \nabla_y H(x(t), y(t))\, y'(t) \equiv 0.$$

$$\Rightarrow \quad H(x(t), y(t)) \equiv \text{const.}$$

$H(x, y)$ can be regarded as a Lyapunov candidate for the Hamiltonian equation. If $\lim_{\|(x,y)\|\to\infty} H(x, y) = \infty$, then the level set $\{(x, y) : H(x, y) = c\}$ is closed and bounded.

We conclude that $I_{\max} = (-\infty, +\infty)$. Otherwise, there exists a time $\omega_+ < \infty$ ($\omega_- > -\infty$) s.t. $\overline{\lim}_{t \to \omega_+^-(\omega_-^+)} \|(x(t), y(t))\| = \infty$, Which yields that there exist

$$(x(t_n), y(t_n)) = (x_n, y_n) \in \{(x, y) : H(x, y) = c\} \text{ s.t. } \lim_{\|(x_n, y_n)\|\to\infty} H(x_n, y_n) = \infty .$$

However, it is not possible. This completes the proof. □

3) Van der Pol Equation

$$x'' = \varepsilon(1 - x^2)x' - x$$

is called Van der Pol equation, where $\varepsilon > 0$ is a small parameter. The form of system:

$$\begin{cases} x' = y \\ y' = \varepsilon(1 - x^2)y - x \end{cases},$$

can be regarded as a perturbation of a particular Hamiltonian system:

$$\begin{cases} x' = y \\ y' = -x \end{cases}.$$

By which we find $H(x, y) = \frac{1}{2}(x^2 + y^2)$ satisfying $\lim_{\|x\| \to \infty} H(x, y) = \infty$, which can be taken as a Lyapunov candidate for the Van der Pol equation. Then we have

$$\frac{d}{dt}H(x, y) = \frac{\partial H}{\partial x}x' + \frac{\partial H}{\partial y}y' = \varepsilon(1 - x^2)y^2 \le \begin{cases} 0, & x^2 \ge 1 \\ \varepsilon y^2, & x^2 \le 1 \end{cases} \le 2\varepsilon H(x, y).$$

By Corollary 5.1, we obtain the global existence. □

4) Dissipative Systems

Let $f : R^n \to R^n$ be locally Lipschitz. Suppose that there exist $v \in R^n$, and $a > 0$, $b > 0$ s.t.

$$\langle f(x), \quad x - v \rangle \le a - b \| x \|^2 .$$

Then the IVP $x' = f(x)$, $x(0) = x_0$, has a unique solution $x(t)$ for $t \ge 0$.

Proof. Taking a ball $B_0 = \{x \in R^n : \| x \|^2 \le \frac{a}{b}\}$ and a Lyapunov candidate as follows.

$$V(x) = \frac{1}{2} \| x - v \|^2$$

satisfying $\lim_{\|x\| \to \infty} V(x) = \infty$, we have

$$\frac{d}{dt} V(x(t)) = \langle f(x), \quad x - v \rangle \le a - b \| x \|^2 < 0 \quad \text{as} \quad \| x \|^2 > \frac{a}{b} .$$

This implies the global existence by Corollary 5.1.

Remark 5.4 The general definition of dissipative systems for $x' = f(x)$ is given as follows. If there exists a bound $B > 0$ s.t. for any solution $x(t)$ of $x' = f(x)$, $x(0) = x_0$, there exists a sufficiently large constant $T(x_0) > 0$, s.t

$$t \geq T(x_0) \quad \Rightarrow \quad \| x(t) \| < B,$$

then $x' = f(x)$ is called a dissipative system. Obviously, the above system is dissipative.

5) Lorentz Equations

The Lorentz equations are given by

$$\begin{cases} x_1' = -\sigma x_1 + \sigma x_2 \\ x_2' = -x_1 x_3 + r x_1 - x_2 \\ x_3' = x_1 x_2 - b x_3 \end{cases},$$

where $\sigma > 0$, $r > 0$ and $b > 1$ are system parameters. (Note: when $r > r_0 = 24.74$, it would exhibit chaotic behavior)

Taking $v = (0, \quad 0, \quad \gamma)$, where $\gamma = \sigma + r$, we have

$$\langle f(x), x - v \rangle = -\sigma x_1^2 - x_2^2 - b x_3^2 + (\sigma + r - \gamma) x_1 x_2 + b\gamma x_3$$

$$= -\sigma x_1^2 - x_2^2 - b x_3^2 + b\gamma x_3 \le -\sigma x_1^2 - x_2^2 - \frac{b}{2} x_3^2 + b\frac{\gamma^2}{2}$$

$$= a - b(x_1^2 + x_2^2 + x_3^2) = a - b\|x\|^2,$$

where $a = b\frac{\gamma^2}{2}$ and $b = \min\{\sigma, 1, \frac{b}{2}\}$. So the Lorentz equations are dissipative.

Comparison Principle

1) Dini Derivative

$$D^{+}v(t)=\overline{\lim_{h\to 0^{+}}}\frac{v(t+h)-v(t)}{h},$$

where $v:R\to R$. When the right limit is unique, we have a right hand derivative as follows.

$$D_{r}v(t)=\lim_{h\to 0^{+}}\frac{v(t+h)-v(t)}{h}.$$

2) Comparison Lemma

Lemma 5.2 (Comparison Lemma) Consider the scalar function $f(t,u)$ is continuous and locally Lipschitz, where $t\geq t_0$ and $u\in R$. If

$$u'(t)=f(t,u(t)),\ u(t_0)=u_0;$$

$$D_{r}v(t)\leq f(t,v(t)),\ v(t_0)=v_0$$

with $v_0\leq u_0$, then $v(t)\leq u(t)$ on any compact interval $t\in[t_0,b]$. **(Homework-1)**

Example 5.1 Find the bound of solution for the IVP $x' = f(x) = -(1+x^2)x$, $x(0) = a$ without solving the equation.

Solution. There exists a unique solution on $[0, \omega_+)$ for some certain $\omega_+ > 0$ (ω_+ could be infinite) because $f(x)$ is continuous and local Lipschitz.

Let $v(t) = x^2(t)$. Then $v(t) \in C^1$ and $v'(t) = 2x(t)x'(t) = -2x^2(t) - 2x^4(t) \le -2x^2(t)$. Hence,

$$v'(t) \le -2v(t), \quad v(0) = a^2.$$

Now consider the IVP $u' = -2u$, $u(0) = a^2 \quad \Rightarrow \quad u(t) = a^2 e^{-2t}$. Then, by the comparison lemma, the solution $x(t)$ is defined on any compact interval $[0, b] \subset [0, \omega_+)$, and satisfies

$$|x(t)| = \sqrt{v(t)} \le e^{-t} |a|, \quad t \in [0, b].$$

First, we say that the above inequality holds for $[0, \omega_+)$ by the extensibility theorem. Then we conclude that the inequality holds for all $t \geq 0$. If it were not the case, it would be time $\omega_+ < \infty$ s.t. $\overline{\lim_{t \to \omega_+^-}} |x(t)| = \infty$ by the extensibility theorem. However, this is not possible because $|x(t)| \leq e^{-\omega_+} |a| < \infty$ for all $t \geq 0$. Therefore,

$$|x(t)| = \sqrt{v(t)} \leq e^{-t} |a|, \quad \forall t \geq 0. \quad \square$$

Example 5.2 Find the bound of solution for the IVP $x' = f(t, x) = -(1 + x^2)x + e^t$, $x(0) = a$ without solving the equation. (**Homework-2**)

3) An Important Lemma for a Vector Function

Lemma 5.3 Suppose that $x(t) \in C^1([a,b])$ is an n-vector valued function, then $D_r(\| x(t) \|)$ exists on $a \le t < b$ and $D_r(\| x(t) \|) \le \| x'(t) \|$, $a \le t < b$.

(Homework-3)

Remark 5.5 This lemma shows that once $D_r(\| x(t) \|)$ exists, derivative sign and norm sign can be exchanged with the inequality relation $D_r(\| x(t) \|) \le \| x'(t) \|$. It looks seemly nothing to do with ODE. However, it is extremely important for ODE with the bound estimation of solution.

4) Comparison Theorem for the Global

Theorem 5.3 (Comparison Theorem) Suppose that $f(t,x)$ of the IVP is continuous and locally Lipschitz, where $t \in [t_0, \omega^+)$ (ω^+ could be infinite) and $x \in R^n$; and satisfies

$$\| f(t,x) \| \le F(t, \| x \|), \quad (t,x) \in [t_0, \omega^+) \times R^n,$$

and $\| x(t_0) \| \le \eta$, where the IVP of the scalar equation

$$u' = F(t,u), \quad u(t_0) = \eta$$

has a unique solution $u(t)$ for $t \in [t_0, \omega^+)$. Then, $x(t)$ exists on $t \in [t_0, \omega^+)$ and

$$\| x(t) \| \le u(t) \quad \text{for all} \quad t \in [t_0, \omega^+).$$

Proof. Let $v(t) = \| x(t) \|$. Then

$$D_r v(t) = D_r \| x(t) \| \le \| x'(t) \| = \| f(t, x(t)) \| \le F(t, \| x(t) \|) = F(t, v(t))$$

and $v(t_0) = \| x(t_0) \| \le \eta$. Application of Lemma 5.1 (**the comparison lemma**) yields

$$\| x(t) \| \le u(t)$$

for any compact interval of $[t_0, \omega^+)$. We conclude that $\| x(t) \| \le u(t)$ for all $t \in [t_0, \omega^+)$. Show by contradiction. If it were not the case, it would be a time c with $t_0 < c < \omega^+$ s.t. $\overline{\lim_{t \to c^-}} \| x(t) \| = \infty$ by the extensibility theorem. But it is not possible because $\| x(c) \| \le u(c) < \infty$. □

Remark 5.6 The result of Theorem 5.3 (**Comparison Theorem**) is global!! It doesn't matter if Lipschitz condition is not satisfied. However, the uniqueness of solution is not guaranteered.

Remark 5.7 Finding $u(t)$ is a key in application of this comparison theorem. For example, $F(t,u)=au+b$ (**Linear Equation**); $F(t,u)=au+bu^n$ (**Bernoulli Equation**); $F(t,u)=g(t)F(u)$ (**Wintner Theorem**) and the others (**DIY**), $u(t)$ can be solved.

5) Some Important Applications

Theorem 5.4 (Wintner Theorem) Suppose that in Theorem 5.3, if

$$\| f(t,x)\| \le g(t)\, L(\| x\|),$$

where $g(t)\ge 0$ is continuous for $t\ge t_0$ and $L(u)\ge 0$ is continuous for $u>0$, and satisfies

$$\int_{u_0}^{+\infty}\frac{du}{L(u)}=+\infty,$$

then the solution $u(t)$ of $u'=g(t)\,F(u)$, $u(t_0)=u_0>0$, with $\| x(t_0)\|=u_0$ exists for all $t\ge t_0$ and satisfies $\| x(t)\|\le u(t)$ for all $t\ge t_0$.

Proof. By Comparison Theorem, we only need to show the existence of $u(t;t_0,u_0)$ for all $t \ge t_0$. Since $u(t)$ satisfies

$$\int_{u_0}^{u} \frac{du}{L(u)} = \int_{t_0}^{t} g(s)ds ,$$

if $u(t)$ would not exist globally on $t \ge t_0$, there would be a finite escape. Then there exists $\omega_+ < \infty$ and $\{t_n\}$ s.t. $\lim_{t_n \to \omega_+^-} u(t_n) = \infty$. That is,

$$\int_{u_0}^{u(t_n)} \frac{du}{L(u)} = \int_{t_0}^{t_n} g(s)ds .$$

But, this is not possible because the left is ∞ and the right is finite. $\square$

Theorem 5.5 For linear equations $x' = A(t)x + h(t)$, where $A(t), h(t) \in C(R)$, then $I_{\max} = [t_0, +\infty)$, $t_0 \in R$.

Proof. In fact,

$$\| A(t)x + h(t) \| \le \| A(t) \| \, \| x \| + \| h(t) \|$$

$$\le \max\{\| A(t) \|, \| h(t) \|\}(\| x \| + 1) = g(t)\, L(\| x \|).$$

Since $L(u) = u + 1$ is continuous and locally Lipschitz, and $\int_0^u \frac{du}{u+1} = \infty$, we have the desired result by Wintner Theorem. □

Remark 5.8 You can prove Theorem 5.5 with $I_{\max} = (-\infty, +\infty)$ by Gronwall inequality and the extensibility theorem. (**Homework-4**)

Summary

- **We introduced three main methods for global existence. They are the linear bounded, Lyaponove method and comparison method.**

- **Several important classes of systems have global existence.**

Homework

1) **Do Homework-1, 2, 3, 4.**
2) **Review today's class.**


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